

Quantization Ambiguity and Supersymmetric Ground State Wave Functions

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Abstract

Supersymmetric ground state wave functions of a model of supersymmetric quantum mechanics on S^1 (supersymmetric simple pendulum) is studied. Supersymmetry can be broken due to existence of an undetermined parameter, which is interpreted as a gauge field and is appeared as a firm consequence of quantization on a space with nontrivial topology like S^1 . The breaking does not depend on leading term of superpotential, contrary to the usual case. The mechanism of the supersymmetry breaking is similar to that through boundary conditions of fields in supersymmetric quantum field theory on compactified space. The supersymmetric harmonic oscillator is realized in the limit of infinite radius of S^1 with the strength of oscillator being constant.

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1 Introduction

In this paper we shall study how quantization ambiguity, which implies that quantization on space with nontrivial topology such as S^1 inevitably yields an undetermined parameter into the theory [1], affects the supersymmetric ground state wave functions of a model of supersymmetric quantum mechanics on S^1 .

Quantization on S^1 is much different from that on one-dimensional euclidean space. In the later case, the representation of the canonical algebra is uniquely determined up to unitary equivalent representation. There is essentially one quantum mechanics on the space. In the former case, however, there is an infinite number of inequivalent representations of the fundamental algebra, which is introduced by Ohnuki-Kitakado [2] in order to formulate quantum mechanics on S^1 . As a result, there exists various quantum mechanics on S^1 .

The various quantum mechanics on S^1 is parametrized by the undetermined parameter. The parameter is interpreted as a constant gauge field [2][3]. The gauge field can exist and has effects on observables at the quantum-level. It is the existence of the gauge field that leads to the various quantum mechanics on S^1 . It may be interesting to study the possible effects of the gauge field on the supersymmetric ground state (zero-energy state) wave functions of supersymmetric quantum mechanics on S^1 . The zero-energy state wave functions can be obtained in closed form [4] because, thanks to the supersymmetry algebra satisfied by the system, the wave functions are obtained by solving simple first-order equations in many cases. Therefore, it may be possible to study the effects as analytically as possible.

Supersymmetric quantum mechanics has been studied in great detail and applied to many physics fields [5]. Actually, it provides us an example of the dynamical supersymmetry breaking by instantons in certain models [6][7][8][9]. In those models, the normalizability of the supersymmetric ground state wave function crucially depends on the leading term in the superpotential, by which we determine whether or not the supersymmetry is broken. The semi-classical instanton approximation has been used to estimate the ground-state energy for the system with broken supersymmetry¹.

In this paper we will find another mechanism of supersymmetry breaking. The very existence of the gauge field twists boundary conditions of supersymmetric ground state wave functions. For certain values of the gauge field, the wave functions do not satisfy a required periodic boundary condition and become unphysical though they are normalizable. The supersymmetry breaking does not depend on the structure of superpotential, unlike the usual supersymmetry breaking discussed in supersymmetric quantum

¹Estimating ground-state energy is a subtle problem. Actually, it is reported in [10][11] that the instanton calculation is very limited though it gives us excellent estimation in some cases and the breakdown of the symmetry is caused by an interplay of perturbative and nonperturbative effects.

mechanics. Supersymmetry can be broken by the gauge field, that is, the quantization ambiguity.

In the next section we shall introduce a model of supersymmetric quantum mechanics on S^1 after reviewing briefly quantum mechanics on S^1 formulated by Ohnuki-Kitakado. And then, we shall discuss how the gauge field affects the supersymmetric ground state wave functions and how it yields the supersymmetry breaking. We shall also study an infinite limit of the radius of S^1 . The supersymmetric harmonic oscillator is realized in the limit with the strength of oscillator being constant. The final section is devoted to conclusions and discussions, where we shall also discuss similarities between our mechanism of supersymmetry breaking and that through boundary conditions of fields in supersymmetric quantum field theory on compactified space.

2 Supersymmetric Quantum Mechanics on S^1

We shall study effects of quantization ambiguity, which implies that quantization on a space with nontrivial topology yields an undetermined parameter, on supersymmetric ground state wave functions of supersymmetric quantum mechanics on S^1 . Let us consider a system in which there are fermionic operator \hat{Q}_i that commutes with the hamiltonian \hat{H} and satisfies the supersymmetry algebra

$$[\hat{Q}_i, \hat{H}] = 0, \quad \{\hat{Q}_i, \hat{Q}_j\} = \delta_{ij} \hat{H} \quad i = 1, \dots, N. \quad (1)$$

$N = 2$ is the simplest case and it is of our interest.

Since the hamiltonian is positive semi-definite, a supersymmetric state $\hat{Q}_i|\Psi\rangle = 0$ is automatically a zero-energy ground state. Conversely, if we have a zero-energy state, it has to be a supersymmetric ground state. Thanks to this property, finding supersymmetric ground states are reduced to solving simple first-order equations instead of solving the second order equation $\hat{H}|\Psi\rangle = 0$. A key point for our study is that the fermionic operator $\hat{Q}_i (i = 1, 2)$ should be written in terms of the operators which are appropriate to describe quantum mechanics on S^1 as shown in the subsection **2.2**. And we shall study the supersymmetric ground state wave functions of such the system.

2.1 Quantum Mechanics on S^1

Before we proceed to a model of supersymmetric quantum mechanics on S^1 , it may be important and instructive to review briefly Ohnuki-Kitakado's formulation of quantum mechanics on S^1 [2]. Those who are familiar with their formulation can skip this subsection and go directly to the subsection **2.2** where the supersymmetric quantum mechanics on S^1 is introduced. Discussions below are based on a paper [3] in a part of which quantum mechanics on S^1 is summarised clearly.

Quantum mechanics on S^1 is defined by a self-adjoint operator \hat{G} and a unitary operator \hat{W} satisfying the commutation relation

$$[\hat{G}, \hat{W}] = \hbar \hat{W}. \quad (2)$$

The operators \hat{G} , \hat{W} and \hat{W}^\dagger generate an algebra. Let us construct its representation. We shall start with an eigenvalue equation

$$\hat{G}|\alpha\rangle = \hbar \alpha |\alpha\rangle \quad \text{with} \quad \langle \alpha | \alpha \rangle = 1, \quad (3)$$

where an eigenvalue α is a real number. It is easy to see that $\hat{W}(\hat{W}^\dagger)$ raises (lowers) the eigenvalues of \hat{G}

$$\hat{G}\hat{W}|\alpha\rangle = \hbar(\alpha + 1)\hat{W}|\alpha\rangle, \quad \hat{G}\hat{W}^\dagger|\alpha\rangle = \hbar(\alpha - 1)\hat{W}^\dagger|\alpha\rangle. \quad (4)$$

A state vector defined by

$$|n + \alpha\rangle \equiv \hat{W}^n |\alpha\rangle, \quad n = \text{integer}. \quad (5)$$

is also an eigenstate of \hat{G}

$$\hat{G}|n + \alpha\rangle = \hbar(n + \alpha)|n + \alpha\rangle. \quad (6)$$

For fixed α , our Hilbert space, denoted by \mathcal{H}_α where the two operators \hat{G} , \hat{W} are defined, is given by completing the vector space of linear combinations of $|n + \alpha\rangle (n = 0, \pm 1, \pm 2, \dots)$. The set of state vectors forms the orthocomplete system in \mathcal{H}_α . Therefore, we have

$$\langle m + \alpha | n + \alpha \rangle = \delta_{mn}, \quad \sum_{n=-\infty}^{+\infty} |n + \alpha\rangle \langle n + \alpha| = \mathbf{1}_\alpha, \quad (7)$$

where $\mathbf{1}_\alpha$ is an identity operator in \mathcal{H}_α . Eq.(6) and $\hat{W}|n + \alpha\rangle = |n + 1 + \alpha\rangle$ define an irreducible representation of the algebra (2) on \mathcal{H}_α . The classification of the irreducible representation of the algebra may be done by noting that (i) \mathcal{H}_α and \mathcal{H}_β are unitary equivalent Hilbert space if and only if $\alpha - \beta = \text{integer}$ (ii) for an arbitrary irreducible representation \mathcal{H} of the algebra, there exists a real number α such that \mathcal{H} is unitary equivalent to \mathcal{H}_α . Thus, the classification is completed, that is, all the inequivalent irreducible representation is given by the Hilbert space $\mathcal{H}_\alpha (0 \leq \alpha < 1)$. It should be emphasized that the algebra (2) has an infinite number of inequivalent representations characterized by an undetermined parameter α , as contrary to the usual irreducible representation of the canonical algebra on one-dimensional euclidean space.

So far, we have constructed the \hat{G} -diagonal representation. One can also go to \hat{W} -diagonal representation by which we will obtain wave functions in quantum mechanics on S^1 . For fixed representation space \mathcal{H}_α , since \hat{W} is a unitary operator, the eigenvalue equation for it may be written as

$$\hat{W}|\theta\rangle = e^{i\theta}|\theta\rangle. \quad (8)$$

Its solution is

$$|\theta\rangle = \kappa(\theta) \sum_{n=-\infty}^{+\infty} e^{-in\theta} |n + \alpha\rangle, \quad (9)$$

where θ is a real parameter and $\kappa(\theta)$ is an arbitrary complex-valued function satisfying $|\kappa(\theta)| = 1$ and $\kappa(\theta + 2\pi) = \kappa(\theta)$. It is not difficult to show

$$|\theta + 2\pi n\rangle = |\theta\rangle, \quad n = \text{integer}, \quad (10)$$

$$\langle\theta|\theta'\rangle = 2\pi \sum_{n=-\infty}^{n=+\infty} \delta(\theta - \theta' + 2\pi n), \quad (11)$$

$$\int_0^{2\pi} \frac{d\theta}{2\pi} |\theta\rangle\langle\theta| = \sum_{n=-\infty}^{+\infty} |n + \alpha\rangle\langle n + \alpha| = \mathbf{1}_\alpha, \quad (12)$$

$$\exp(-i\lambda \frac{\hat{G}}{\hbar})|\theta\rangle = e^{-i\lambda\alpha} \kappa(\theta) \kappa^*(\theta + \lambda) |\theta + \lambda\rangle, \quad (13)$$

where $\mathbf{1}_\alpha$ is an identity operator in \mathcal{H}_α . These correspond to periodicity, orthonormality, completeness and translation for the eigenstate of \hat{W} . Let us note that it may be possible from (8) and (13) to identify \hat{G} and \hat{W} with the momentum and the position operators on S^1 , respectively.

Now, let $|\psi\rangle$ be a state vector and we define a wave function $\psi(\theta)$ on S^1 as follows

$$\psi(\theta) \equiv \langle\theta|\psi\rangle. \quad (14)$$

Taking inner product of Eq. (13) with $|\psi\rangle$, we obtain

$$\langle\theta|\exp(i\lambda \frac{\hat{G}}{\hbar})|\psi\rangle = e^{i\lambda\alpha} \kappa^*(\theta) \kappa(\theta + \lambda) \langle\theta + \lambda|\psi\rangle, \quad (15)$$

from which the \hat{W} -diagonal representation of \hat{G} is given by

$$\langle\theta|\hat{G}|\psi\rangle = \left[-i\hbar \frac{\partial}{\partial\theta} - i\hbar \kappa^*(\theta) \frac{\partial\kappa(\theta)}{\partial\theta} + \hbar\alpha \right] \psi(\theta). \quad (16)$$

We also obtain from (8)

$$\langle\theta|\hat{W}|\psi\rangle = e^{i\theta} \psi(\theta). \quad (17)$$

The inner product on S^1 is expressed in terms of the wave function as

$$\langle\chi|\psi\rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} \chi^*(\theta) \psi(\theta). \quad (18)$$

Thus, the representation of the Hilbert space, which is defined by Eqs. (16) and (17), is the space of square integrable function on S^1 . Let us note that all wave functions have to satisfy the periodic boundary condition $\psi(\theta + 2\pi n) = \psi(\theta)$, which is a direct consequence of (10). This periodicity is essential when we study the supersymmetric ground state wave functions of the supersymmetric quantum mechanics on S^1 .

Let us next present the physical meaning of the parameter α . To this end, let us redefine $\kappa(\theta)$ by utilizing the arbitrariness of it in such a way that $\kappa(\theta) = \omega(\theta) \kappa'(\theta)$,

where $\omega(\theta)$ has to satisfy $|\omega(\theta)| = 1$ and $\omega(\theta + 2\pi) = \omega(\theta)$. It follows $|\theta\rangle = \omega(\theta)|\theta'\rangle$, so that the transformed wave function $\psi'(\theta)$ is given by

$$\psi'(\theta) = \omega(\theta)\psi(\theta). \quad (19)$$

According to this redefinition, the \hat{W} -diagonal representation for \hat{G} becomes

$$'\langle\theta|\hat{G}|\psi\rangle = \left[-i\hbar\frac{\partial}{\partial\theta} + A'(\theta)\right]\psi(\theta), \quad (20)$$

where we have defined

$$A'(\theta) \equiv A(\theta) + i\hbar\omega^*(\theta)\frac{\partial\omega(\theta)}{\partial\theta} \quad \text{and} \quad A(\theta) \equiv -i\hbar\kappa^*(\theta)\frac{\partial\kappa(\theta)}{\partial\theta} + \hbar\alpha. \quad (21)$$

Eqs.(19) and (21) stand for the gauge transformation. Therefore, the parameter α has a meaning of the gauge field. It is easy to see that the gauge field has properties (i) $A(\theta)$, assumed to be an arbitrary real-valued function satisfying periodic boundary condition $A(\theta + 2\pi) = A(\theta)$, can always be made a constant function $A'(\theta) = \alpha$ by a gauge transformation (ii) for two constant functions $A'(\theta) = \alpha$ and $A'(\theta) = \beta$, these are connected by a unique gauge transformation if and only if $\beta - \alpha$ is an integer. Thus, we arrive at an important conclusion that all the inequivalent gauge field is given by $A_\alpha \equiv \alpha(0 \leq \alpha < 1)^2$. Hereafter, we choose $\kappa(\theta) = 1$ for simplicity.

It is a very special feature of quantum mechanics on S^1 that the inequivalent gauge field is restricted to be $0 \leq \alpha < 1$. Another way of looking at it is that if we perform a gauge transformation by $\psi(\theta) \rightarrow \psi'(\theta) = e^{in\theta}\psi(\theta)$, we see that the gauge field $A(\theta)$ and $A(\theta) - n\hbar$ are equivalent for $n = \text{integer}$. n has to be restricted to be integer, otherwise the transformed wave function $\psi'(\theta)$ does not satisfy the required periodic boundary condition. Therefore, the inequivalent gauge field is given by $0 \leq A(\theta) < \hbar$, which means $0 \leq \alpha < 1$. Let us note that the gauge transformation by $e^{in\theta}$ with $n = \text{non-integer}$ is a singular gauge transformation and is strictly forbidden.

Different values of the gauge field give different quantum mechanics on S^1 . It may be helpful to note that the gauge field α may correspond to the magnitude of the magnetic flux $e\Phi/2\pi\hbar c$ through S^1 in the Aharonov-Bohm effect. The different magnitude of the flux actually gives different physics.

2.2 Supersymmetric Simple Pendulum

Now, we are ready to introduce a model of supersymmetric quantum mechanics on S^1 . According the discussion above, the two operators \hat{G} and \hat{W} , which correspond to the

²Dirac's approach to the quantization for a constrained system does not yield an infinite number of the inequivalent representations, that is, it corresponds to only $\alpha = 0$. Quantization and embedding S^1 into higher dimensional space \mathbf{R}^2 is not a "commutative" procedure to each other. Let us also note that the Ohnuki-Kitakado formulation is independent of the dynamics, in contrast to the Dirac's approach.

momentum and the position of a particle on S^1 , are fundamental. It may be natural to construct a quantum hamiltonian in terms of these operators. The hamiltonian in our model is assumed to satisfy the supersymmetry algebra (1), so that the fermionic operators $\hat{Q}_i (i = 1, 2)$ also has to be given in terms of them. We will discuss the classical counterpart of the quantum hamiltonian constructed in this way later.

Let us define the fermionic operator $\hat{Q}_i (i = 1, 2)$ by

$$\begin{aligned}\hat{Q} &\equiv \frac{1}{\sqrt{2}}(\hat{Q}_1 + i\hat{Q}_2) \\ &= \left(\frac{1}{\sqrt{2mR}}\hat{G} + iV(\hat{W}, \hat{W}^\dagger)\right)\hat{\xi} \equiv \hat{q}\hat{\xi},\end{aligned}\tag{22}$$

$$\begin{aligned}\hat{\bar{Q}} &\equiv \frac{1}{\sqrt{2}}(\hat{Q}_1 - i\hat{Q}_2) \\ &= \left(\frac{1}{\sqrt{2mR}}\hat{G} - iV(\hat{W}, \hat{W}^\dagger)\right)\hat{\bar{\xi}} \equiv \hat{q}^\dagger\hat{\bar{\xi}}.\end{aligned}\tag{23}$$

$V(\hat{W}, \hat{W}^\dagger)$, which is called superpotential hereafter, is a hermitian operator in terms of \hat{W} and \hat{W}^\dagger . m and R stand for mass of a particle and the radius of S^1 , respectively. The fermionic variables $\hat{\xi}, \hat{\bar{\xi}}$ satisfy the algebra

$$\{\hat{\xi}, \hat{\bar{\xi}}\} = 1, \quad \hat{\xi}^2 = \hat{\bar{\xi}}^2 = 0.\tag{24}$$

Then, the hamiltonian is given by

$$\begin{aligned}\hat{H} &= \{\hat{Q}, \hat{\bar{Q}}\} \\ &= \frac{1}{2mR^2}\hat{G}^2 + V^2(\hat{W}, \hat{W}^\dagger) \\ &\quad - \frac{i}{\sqrt{2mR}}(\hat{G}V(\hat{W}, \hat{W}^\dagger) - V(\hat{W}, \hat{W}^\dagger)\hat{G})[\hat{\xi}, \hat{\bar{\xi}}],\end{aligned}\tag{25}$$

where we have used (24).

Here, it may be necessary to discuss the classical counterpart of the quantum hamiltonian (25). To this end, let us note that the fundamental algebra (2) may be actually inferred by the classical Poisson's bracket for the angle variable θ and the corresponding momentum P_θ in the polar coordinate

$$\{P_\theta, e^{i\theta}\}_P = -ie^{i\theta}.\tag{26}$$

If we replace the classical Poisson's bracket by the commutation relation divided by $i\hbar$, we obtain the fundamental algebra (2) by identifying $e^{i\theta}$ and P_θ with \hat{W} and \hat{G} , respectively. This is the same identification stated earlier. Therefore, in the classical limit we may replace \hat{W} by $e^{i\theta}$ and \hat{G} by P_θ . According to these replacements, we obtain a classical hamiltonian, ignoring the fermionic variables $\hat{\xi}, \hat{\bar{\xi}}$

$$\hat{H} \rightarrow H_{cl} = \frac{P_\theta^2}{2mR^2} + V^2(e^{i\theta}, e^{-i\theta}).\tag{27}$$

If we choose $V(e^{i\theta}, e^{-i\theta})$ as

$$V(e^{i\theta}, e^{-i\theta}) = \sqrt{\frac{mg_N R}{2}} \sin \theta, \quad (28)$$

the classical hamiltonian describes a simple pendulum with angle 2θ . g_N is the gravitation accelerator constant.

On the other hand, given the superpotential (28), the quantum counterpart of it is obtained by

$$V(\hat{W}, \hat{W}^\dagger) = \sqrt{\frac{mg_N R}{2}} \left(\frac{\hat{W} - \hat{W}^\dagger}{2i} \right). \quad (29)$$

Having this superpotential, the model of supersymmetric quantum mechanics on S^1 , that is, *supersymmetric simple pendulum* is given by the hamiltonian

$$\hat{H} = \frac{1}{2mR^2} \hat{G}^2 + \frac{mg_N R}{2} \sin^2 \theta - \frac{\hbar}{2} \sqrt{\frac{g_N}{R}} \cos \theta [\hat{\xi}, \hat{\xi}] \quad (30)$$

in the \hat{W} -diagonal representation. It is understood that $\hat{G} = -i\hbar \partial/\partial\theta + \hbar\alpha$ and α is the gauge field discussed in the previous subsection.

Let us study the supersymmetric ground state wave functions of the supersymmetric simple pendulum whose hamiltonian is given by (30). It is followed from the algebra (1) that the supersymmetric ground states must be zero-energy states satisfied by

$$\hat{Q}|\Psi\rangle = 0 \quad \text{and} \quad \hat{\bar{Q}}|\Psi\rangle = 0. \quad (31)$$

Let us introduce a matrix representation for the fermionic variables. It is easy to see that the matrix representations given by

$$\hat{\xi} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \hat{\bar{\xi}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (32)$$

satisfy the algebra (24). Then, it follows

$$[\hat{\xi}, \hat{\bar{\xi}}] = - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv -\sigma^3.$$

Using these, the hamiltonian (30) becomes

$$\begin{aligned} \hat{H} &= \left(\frac{1}{2mR^2} \hat{G}^2 + \frac{mg_N R}{2} \sin^2 \theta \right) \mathbf{1}_{2 \times 2} + \frac{\hbar}{2} \sqrt{\frac{g_N}{R}} \sigma^3 \cos \theta \\ &= \begin{pmatrix} \hat{q}^\dagger \hat{q} & 0 \\ 0 & \hat{q} \hat{q}^\dagger \end{pmatrix} \equiv \begin{pmatrix} \hat{H}_+ & 0 \\ 0 & \hat{H}_- \end{pmatrix}. \end{aligned} \quad (33)$$

In the matrix representation, the hamiltonian is a 2×2 matrix.

Since the hamiltonian (33) commutes with an operator $\hat{S}^F \equiv \sigma^3/2$, the eigenstates of the hamiltonian is labelled by the eigenvalues of \hat{S}^F . Let us call the two states $|+\rangle$ and $|-\rangle$

fermion number $+1/2$ and $-1/2$, respectively³. The state vector is, now, a two-component vector

$$|\Psi\rangle = \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}. \quad (34)$$

In the \hat{W} -diagonal representation, it may be written as

$$\Psi(\theta) = \begin{pmatrix} \psi_{+\frac{1}{2}}(\theta) \\ \psi_{-\frac{1}{2}}(\theta) \end{pmatrix}. \quad (35)$$

The hamiltonian \hat{H} is diagonalized with respect to the fermion number $\pm 1/2$. In this matrix representation, (31) is read as

$$\begin{aligned} \hat{q} \psi_{+\frac{1}{2}}(\theta) &= \left(\frac{1}{\sqrt{2mR}} (-i\hbar \frac{\partial}{\partial \theta} + \hbar \alpha) + i\sqrt{\frac{mg_N R}{2}} \sin \theta \right) \psi_{+\frac{1}{2}}(\theta) = 0, \\ \hat{q}^\dagger \psi_{-\frac{1}{2}}(\theta) &= \left(\frac{1}{\sqrt{2mR}} (-i\hbar \frac{\partial}{\partial \theta} + \hbar \alpha) - i\sqrt{\frac{mg_N R}{2}} \sin \theta \right) \psi_{-\frac{1}{2}}(\theta) = 0. \end{aligned} \quad (36)$$

Solutions for Eq. (36) are found to be

$$\psi_{+\frac{1}{2}}(\theta) = \frac{1}{\sqrt{I_0(2z)}} \exp(-i\alpha\theta - \frac{z}{\hbar} \cos \theta), \quad \psi_{-\frac{1}{2}}(\theta) = \frac{1}{\sqrt{I_0(2z)}} \exp(-i\alpha\theta + \frac{z}{\hbar} \cos \theta), \quad (37)$$

where $I_0(2z)$ in the normalization factor is the 0-th order modified Bessel function and we have defined a dimensionless parameter

$$\frac{mR^2}{\hbar} \sqrt{\frac{g_N}{R}} \equiv \frac{mR^2}{\hbar} \omega \equiv \frac{z}{\hbar}.$$

These are normalizable solutions. Thus, one may say that the zero-energy states, that is, supersymmetric ground states, exist in the model and the supersymmetry is unbroken. This is, however, a hasty conclusion. In addition to the normalizability, all the wave function has to satisfy the periodic boundary condition $\Psi(\theta + 2\pi) = \Psi(\theta)$, that is, $\psi_{\pm\frac{1}{2}}(\theta + 2\pi) = \psi_{\pm\frac{1}{2}}(\theta)$, which is followed from Eq. (10)⁴. It is easy to see from (37) that

$$\psi_{\pm\frac{1}{2}}(\theta + 2\pi) = e^{-i2\pi\alpha} \psi_{\pm\frac{1}{2}}(\theta). \quad (38)$$

The boundary condition for the zero-energy state wave functions is twisted by the gauge field α . The zero-energy state wave functions do not satisfy the required periodic boundary condition except for $\alpha = \text{integer}$. Since the inequivalent representation is given by $0 \leq \alpha < 1$, they are inconsistent with the periodic boundary condition and become *unphysical* wave functions for $0 < \alpha < 1$. Therefore, the supersymmetry can be broken due to the

³One may call the two states spin-up and spin-down states. Or by introducing $\hat{f} = \frac{1}{2} + \frac{1}{2}[\hat{\xi}, \hat{\xi}]$ whose eigenvalues are 0, 1, one may denote them by fermion numbers 0, 1.

⁴Let us note that the two-component spinor state (34) does not have the minus sign under 2π -rotation in this case because the rotation is done by the usual rotation matrix in the 2-dimensions.

gauge field α . Let us note that the Witten index $\text{Tr}(-1)^{\hat{f}} = n_B^{E=0} - n_F^{E=0}$ vanishes in our model. It is easy to see that $n_B^{E=0} = n_F^{E=0} = 1$ for $\alpha = \text{integer}$ and that $n_B^{E=0} = n_F^{E=0} = 0$ for $\alpha = \text{non-integer}$.

Unlike the usual supersymmetry breaking, in which the leading term of the superpotential determines whether or not supersymmetry is broken, our breaking of supersymmetry does not depend on the structure of the superpotential. It is entirely due to the existence of the gauge field α , which is an inevitable consequence of the quantization ambiguity when one quantizes the theory on topologically nontrivial space such as S^1 . The gauge field has the effect to twist the boundary conditions of the zero-energy state wave functions. Among the various supersymmetric quantum mechanics on S^1 lead by the gauge field, it includes the theories with the broken supersymmetry due to the gauge field. Let us note that in this context there is no mechanism to determine the values of α , or what values of α we should take.

Note that for non-integer values of the gauge field α , it can not be removed in (36) by the gauge transformation

$$\psi_{\pm\frac{1}{2}}(\theta) \longrightarrow \psi'_{\pm\frac{1}{2}}(\theta) = e^{-i\alpha\theta} \psi_{\pm\frac{1}{2}}(\theta). \quad (39)$$

As we stated before, the inequivalent gauge field is given by $0 \leq \alpha < 1$. Any gauge field in the range can not be connected by the regular gauge transformation with $e^{in\theta}$ ($n = \text{integer}$). Only a singular gauge transformation can do it, but it destroys the required periodic boundary condition for the transformed wave function. The singular gauge transformation is strictly forbidden, so that the gauge field can not be gauged away.

Let us briefly comment on the ground-state energy. We have a physical supersymmetric ground state wave function for $\alpha = \text{integer}$, so that the ground-state energy is exactly zero. On the other hand, for $0 < \alpha < 1$, the supersymmetry is broken. The ground-state energy is non-zero (positive). Estimating the ground-state energy is a subtle problem as studied in [10][11].

The hamiltonian (33) can not be solved analytically. The bosonic potential $\frac{mg_N R}{2} \sin^2 \theta$ is periodic and the classical vacuum has a periodic structure. One may expect that there is a instanton-like classical solution, which gives a finite euclidean action, connecting the two vacua with different fermion number ($\pm 1/2$). Actually, there exists such a classical solution in our model. It is given by $\cos \theta_{cl}(\tau) = \pm \tanh(\omega(\tau - \tau_0))$ with the classical euclidean action being $-2z/\hbar$. And the fermion zero mode exists in this classical background. Therefore, we expect to occur the tunneling between the two vacua. According to the semi-classical argument, the tunneling effect shifts the ground-state energy to give exponentially small amount of the energy in the form of $\exp(-2z/\hbar) \times \cos 2\pi\alpha^5$.

⁵This band structure $\cos 2\pi\alpha$ can be understood from the effective action obtained by the transition amplitude $K(\theta_f, t; \theta_i, 0) = \langle \theta_f | \exp(-i\hat{H}t/\hbar) | \theta_i \rangle = \sum_{n=-\infty}^{+\infty} \int_{n\text{-winding}} \mathcal{D}\theta \exp(iS_{eff}/\hbar)$, where the

On the other hand, for very small z , we can resort to the perturbation theory to obtain the energy spectrum of the hamiltonian. The ground-state energy is given by $E_0 \sim \frac{1}{2mR^2}(\alpha^2 + O(z^2))$, where we have set $\hbar = 1$. The gauge field α is a dominant contribution to the ground-state energy in this case.

It may be interesting to consider $R \rightarrow \infty$ limit. So far, we have fixed the radius R of S^1 . If R varies to become large, we expect that the arc of an arbitrary part of S^1 will approach to a straight segment. In the limit of $R \rightarrow \infty$, one-dimensional euclidean space will be recovered.

In order to study the limit, let us define a variable $x \equiv R\theta$. In terms of this new variable, the hamiltonian (33) is written as, remembering $\hat{G} = -i\hbar \partial/\partial\theta + \hbar \alpha$ in the \hat{W} -diagonal representation,

$$\hat{H} = \left(\frac{-\hbar^2}{2m} \left(\frac{\partial}{\partial x} + i \frac{\alpha}{R} \right)^2 + \frac{mg_N R}{2} \sin^2 \left(\frac{x}{R} \right) \right) \mathbf{1}_{2 \times 2} + \frac{\hbar}{2} \sqrt{\frac{g_N}{R}} \sigma^3 \cos \left(\frac{x}{R} \right). \quad (40)$$

If we take the limit of $R \rightarrow \infty$ naively, the hamiltonian becomes trivial to yield the one for a free particle on S^1

$$\hat{H} = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \mathbf{1}_{2 \times 2}. \quad (41)$$

In order to obtain an interacting theory, one has to take the limit with keeping a relation given by

$$\frac{m}{\hbar} \sqrt{\frac{g_N}{R}} \equiv \frac{m\omega}{\hbar} = \left(\text{strength of oscillator} \right)^2 = \text{constant}. \quad (42)$$

Then, we obtain

$$\hat{H} = \left(\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2} x^2 \right) \mathbf{1}_{2 \times 2} + \frac{\hbar\omega}{2} \sigma^3 + O\left(\frac{1}{R^2}\right). \quad (43)$$

This is the well-known hamiltonian for the supersymmetric harmonic oscillator [6] with the angular frequency $\omega = \sqrt{g_N/R}$. Likewise, by taking the same limit, the fermionic operators $\hat{Q}, \hat{\bar{Q}}$ become

$$\begin{aligned} \hat{Q} &= \frac{1}{\sqrt{2m}} (\hat{p} + iW(\hat{x})) \hat{\xi} + O\left(\frac{1}{R^2}\right) \equiv \hat{Q}_{susy} + O\left(\frac{1}{R^2}\right), \\ \hat{\bar{Q}} &= \frac{1}{\sqrt{2m}} (\hat{p} - iW(\hat{x})) \hat{\bar{\xi}} + O\left(\frac{1}{R^2}\right) \equiv \hat{\bar{Q}}_{susy} + O\left(\frac{1}{R^2}\right), \end{aligned} \quad (44)$$

where we have defined $W(\hat{x}) \equiv m\omega\hat{x}$ and $\hat{p} \equiv -i\hbar \partial/\partial x$. It is easy to check that the hamiltonian (43) satisfies the supersymmetry algebra (1) with the supercharges (44) if we use the canonical commutation relation $[\hat{p}, \hat{x}] = -i\hbar$ and (32). The usual supersymmetry, by which the supersymmetry transformations between boson (\hat{x}) and fermion ($\hat{\xi}, \hat{\bar{\xi}}$) are generated, is realized in the limit of $R \rightarrow \infty$ with (42).

effective action is given by $S_{eff} = \int dt \frac{mR^2}{2} \left(\frac{d\theta}{dt} \right)^2 - \frac{mg_N R}{2} \sin^2 \theta + \frac{1}{2} \sqrt{\frac{g_N}{R}} \cos \theta [\xi, \xi^*] + i\xi^* \frac{d\xi}{dt} - \alpha \frac{d\theta}{dt}$. The “topological” term $\alpha \dot{\theta}$ is the origin of such the band structure[12].

The supersymmetric ground state wave functions for the hamiltonian (43) are obtained by solving the first-order equation $\hat{Q}_{susy}|\Psi\rangle = 0$ and $\hat{\bar{Q}}_{susy}|\Psi\rangle = 0$. Using the same matrix representation as before, their solutions are

$$\psi_{+\frac{1}{2}}^{h.o.}(x) \sim \exp(+\frac{m\omega}{2\hbar}x^2), \quad \psi_{-\frac{1}{2}}^{h.o.}(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp(-\frac{m\omega}{2\hbar}x^2). \quad (45)$$

There are two candidates for the supersymmetric ground state (zero-energy state) wave functions. The one is physical and its wave function is given by $\psi_{-\frac{1}{2}}^{h.o.}(x)$. The other one $\psi_{+\frac{1}{2}}^{h.o.}(x)$ is unphysical because of its non-normalizability. Therefore, we have one supersymmetric ground (zero-energy) state. This is consistent with the exact energy spectrum of the hamiltonian (43). As easily seen from the hamiltonian, there exists one supersymmetric ground state. In fact, the solutions (45) can be obtained by taking the limit of $R \rightarrow \infty$ with the relation (42) in (37). By noting $I_0(z) \sim e^z/\sqrt{2\pi z}$ for large z , we obtain

$$\tilde{\psi}_{+\frac{1}{2}} \sim \exp(+\frac{m\omega}{2\hbar}x^2), \quad \tilde{\psi}_{-\frac{1}{2}}(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp(-\frac{m\omega}{2\hbar}x^2), \quad (46)$$

where we have redefined the normalization as $\tilde{\psi}_{\pm\frac{1}{2}}(x)dx \equiv \psi_{\pm\frac{1}{2}}(\theta)\frac{d\theta}{\sqrt{2\pi R}}$. These are the same with (45). The Witten index is $\text{Tr}(-1)^{\hat{f}} = 1$ in this case.

A simple generalization of the model is a N -component one. The fermionic operators are defined by

$$\hat{Q} = \sum_{a=1}^N \left(\frac{1}{\sqrt{2mR_a}} \hat{G}_a + i\hat{V}_a \right) \hat{\xi}_a \equiv \sum_{a=1}^N \hat{q}_a \hat{\xi}_a, \quad (47)$$

$$\hat{\bar{Q}} = \sum_{a=1}^N \left(\frac{1}{\sqrt{2mR_a}} \hat{G}_a - i\hat{V}_a \right) \hat{\bar{\xi}}_a \equiv \sum_{a=1}^N \hat{q}_a^\dagger \hat{\bar{\xi}}_a, \quad (48)$$

where $\hat{V}_a \equiv \hat{V}_a(\theta_1, \dots, \theta_N)$ and $\hat{G}_a = -i\hbar \partial/\partial\theta_a + \hbar \alpha_a$ in the \hat{W} -diagonal representation. Let us assume

$$[\hat{G}_a, \hat{W}_b] = \hbar \delta_{ab} \hat{W}_b, \quad \{\hat{\xi}_a, \hat{\bar{\xi}}_b\} = \delta_{ab}, \quad \{\hat{\xi}_a, \hat{\xi}_b\} = 0, \quad \{\hat{\bar{\xi}}_a, \hat{\bar{\xi}}_b\} = 0. \quad (49)$$

Then, the hamiltonian followed from these fermionic operators is

$$\hat{H} = \sum_{a=1}^N \frac{1}{2mR_a^2} \hat{G}_a \hat{G}_a + \hat{V}_a \hat{V}_a - \sum_{a,b=1}^N \frac{i}{\sqrt{2mR_a}} [\hat{G}_a, \hat{V}_b] [\hat{\xi}_a, \hat{\bar{\xi}}_b]. \quad (50)$$

The hamiltonian (50) may describe the supersymmetric quantum mechanics on the torus $T^N = S^1 \otimes \dots \otimes S^1$. As before, $\alpha_a (a = 1, \dots, N)$ may be interpreted as the gauge field appeared as the consequence of quantization on the each topological space S^1 .

It is difficult to obtain the exact form of the supersymmetric ground state wave functions of the model. If we, however, restrict ourself to certain sectors of the model, they can be obtained in closed form like (53) [4]. In order to see it, let us define

$$|-\rangle \equiv |0\rangle, \quad |+\rangle \equiv \prod_{a=1}^N \hat{\bar{\xi}}_a |0\rangle \quad \text{with} \quad \hat{\xi}_a |0\rangle = 0 \quad (a = 1, \dots, N). \quad (51)$$

Then, $\hat{Q}|- \rangle = \hat{\bar{Q}}|+ \rangle = 0$ is trivially satisfied, so that in these two sectors, the supersymmetric ground state (zero-energy state) wave functions are obtained in closed form by solving the simple first-order equations like (36). Aside from the normalization, the solutions are obtained as

$$\begin{aligned}\Psi^+(\theta_1, \dots, \theta_N) &= \exp \sum_{a=1}^N \left(-i\alpha_a \theta_a + \frac{\sqrt{2m}R_a}{\hbar} \int^{\theta_a} d\bar{\theta}_a V_a(\bar{\theta}_1, \dots, \bar{\theta}_a, \dots, \bar{\theta}_N) \right) |+\rangle, \\ \Psi^-(\theta_1, \dots, \theta_N) &= \exp \sum_{a=1}^N \left(-i\alpha_a \theta_a - \frac{\sqrt{2m}R_a}{\hbar} \int^{\theta_a} d\bar{\theta}_a V_a(\bar{\theta}_1, \dots, \bar{\theta}_a, \dots, \bar{\theta}_N) \right) |-\rangle.\end{aligned}\quad (52)$$

The wave functions have to satisfy the periodic boundary condition $\Psi^\pm(\dots, \theta_a + 2\pi, \dots) = \Psi^\pm(\dots, \theta_a, \dots)$ ($a = 1, \dots, N$). If the contributions coming from the superpotential in (52) does not spoil the normalizability and the periodicity of the wave functions, the supersymmetry can be broken for non-integer values of α_a ($a = 1, \dots, N$).

3 Conclusions and Discussions

We have applied the Ohnuki-Kitakado's formulation of quantum mechanics on S^1 to the supersymmetric simple pendulum whose hamiltonian is given by (33) and satisfies the algebra (1). According to their formulation, an undetermined parameter, which can be interpreted as a constant gauge field, inevitably enters into theory to yield the various quantum mechanics on S^1 . We have studied the effects of the quantization ambiguity on the supersymmetric ground state wave functions of the model.

We have found that the supersymmetry can be broken due to the existence of the gauge field α . The gauge field twists the boundary condition of the supersymmetric ground state wave functions. For non-integer values of α , they do not satisfy the required periodic boundary condition. As a result, they become unphysical wave functions though they are normalizable. The mechanism of supersymmetry breaking is different from the usual supersymmetry breaking discussed in the supersymmetric quantum mechanics. The latter depends crucially on the structure, leading term, of the superpotential, while the former is entirely due to the quantization ambiguity resulted firmly from quantization on a space with nontrivial topology like S^1 .

We have chosen the superpotential $V(\hat{W}, \hat{W}^\dagger)$ in such a way that it becomes the simple pendulum in the classical limit. In principle, one can choose any superpotential as long as it can be written in terms of integer powers of the operators \hat{W} and \hat{W}^\dagger . Thanks to the factorizable property for finding the supersymmetric ground state wave functions, they are given simply by solving the first order equation (31) and are obtained in closed form

$$\psi_{\pm\frac{1}{2}}(\theta) = \exp\left(-i\alpha\theta \mp \frac{\sqrt{2m}R}{\hbar} \int^\theta d\bar{\theta} V(e^{i\bar{\theta}}, e^{-i\bar{\theta}})\right) \quad (53)$$

for general superpotential. Our mechanism of supersymmetry breaking is not altered by the choice of the superpotential if $\exp(\int^\theta d\bar{\theta} V(e^{i\bar{\theta}}, e^{-i\bar{\theta}}))$ does not violate the periodicity

and the normalizability of the wave functions, which is the case for the superpotential satisfying our criterion. The supersymmetry breaking will always occur for non-integer values of α . Due to the factorizable property, there is no way to prevent the gauge field from entering into the supersymmetric ground state wave functions and twisting their boundary conditions. The gauge field can not be removed by a singular gauge transformation.

One may wonder that all the eigenfunction of the hamiltonian (33) becomes unphysical, that is, those do not satisfy the periodic boundary condition due to the existence of the gauge field. This is not true. In order to see this, let us consider a free hamiltonian, ignoring all the term except for \hat{G}^2 . The energy eigenvalue depend on α like $(m + \alpha)^2$ and the gauge field produces an effect on the observable at the quantum level [3]. The corresponding eigenfunction satisfying the periodic boundary condition is easily found to be $e^{im\theta}$. The ground state wave functions and the other eigenfunction are obtained by solving essentially different types of the differential equations in the system satisfying the supersymmetry algebra (1).

We have also discussed the limit of $R \rightarrow \infty$. One-dimensional euclidean space is realized. In the limit with the relation (42), we have obtained the supersymmetric harmonic oscillator with the angular frequency $\omega = \sqrt{g_N/R}$. There exists one physical supersymmetric ground state wave function. The other one, though it is a zero-energy state, is unphysical because of its non-normalizability. These two wave functions are actually obtained by taking the limit in the solutions of the zero-energy wave functions (37). In the limit all the effect of the gauge field α disappears. Then, an infinite number of the inequivalent representations is reduced to the unique representation, which is nothing but the representation of the canonical algebra $[\hat{p}, \hat{x}] = -i\hbar$. The fermionic operators become the supercharges in the same limit, and they generate the supersymmetry transformations between bosons (\hat{x}) and fermions ($\hat{\xi}, \hat{\bar{\xi}}$).

We have also considered the N -component generalization of (25) and studied the supersymmetric ground state wave functions of the model. The hamiltonian (50) describes the supersymmetric quantum mechanics on the torus T^N . If we restrict ourself to the two sectors given by $|-\rangle$ and $|+\rangle$, the wave functions can be obtained in closed form (52) by solving the simple first-order equations. We have found, again, that the supersymmetry was able to be broken due to the existence of the gauge field $\alpha_a \neq \text{integer}(a = 1, \dots, N)$, which was appeared as the consequence of quantization on the each topological space S^1 .

Finally, let us discuss similarities between our mechanism of supersymmetry breaking and that by boundary conditions of fields for compactified directions in supersymmetric quantum field theory.

Strictly speaking, the supersymmetry breaking through boundary conditions is one thing, and that through our mechanism is another. Nevertheless, it may be interesting to discuss similarities between the two supersymmetry breaking. In the former case, the

breaking means that the action is no longer invariant under the supersymmetry transformations. But it does not necessarily mean non-existence of zero-energy state in the system. The supersymmetric harmonic oscillator at finite temperature is one of the examples, in which the action is not invariant under the supersymmetry transformations because of the different boundary conditions between the bosons and the fermions, there exists, however, the zero-energy state in the system, which is resulted by $\text{Tr}(-1)^{\hat{f}} = 1$ [13]. In the later case, we assume that the hamiltonian satisfies the supersymmetry algebra (1), so that the supersymmetry breaking immediately means non-existence of physical zero-energy states in the system, and whether or not the supersymmetry is broken is determined definitely by the existence of the zero-energy state.

As seen from (38), the boundary condition for the S^1 direction is twisted for $0 < \alpha < 1$ by $e^{-i2\pi\alpha}$. If we consider a theory at finite temperature, it is equivalent to studying the theory in a space where the euclidean time direction is compactified on S^1 . It is well-known that the supersymmetry is broken at finite temperature by the different boundary conditions for the euclidean time direction between the bosons and the fermions. The boson (fermion) satisfies the (anti) periodic boundary condition. The case of $\alpha = 1/2$, which actually corresponds to the anti-periodic boundary condition, is similar to the case of the supersymmetry breaking at finite temperature.

More generally, if one wishes to break the supersymmetry through different boundary conditions between the bosons and the fermions like the finite temperature case, one can use the boundary condition associated with the $U(1)_R$ symmetry [14][15] [16], in which the $U(1)_R$ charges are different between the bosons and the fermions in a supermultiplet. If we regard the factor $e^{-i2\pi\alpha}$ in (38) as a boundary condition that breaks the supersymmetry, our mechanism of supersymmetry breaking is quite similar to that through boundary condition associated with the $U(1)_R$ symmetry. If one takes this similarity seriously, one says that a possible physical origin of the supersymmetry breaking through the boundary condition associated with the $U(1)_R$ symmetry has been found.

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